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On Weak L^1 Convergence of the Densities of the Homogeneous Young Measures

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Example 1 *(O.Bolza, L.C.Young)*

Find the minimum of the functional

$$\mathcal{J}(u) = \int_0^1 \left[u^2 + \left((\tfrac{du}{dx})^2 - 1\right)^2\right] dx,$$

with boundary conditions $u(0) = 0 = u(1)$.

Here $\inf \mathcal{J} = 0$, *but there is no function* u_0 *such that:*

$$u_0 \equiv 0 \text{ a.e. and simultaneously } \tfrac{du_0}{dx} = \pm 1 \text{ a.e..}$$

The minimizing sequence for this functional is of the form

$$u_n(x) = \begin{cases} x - \frac{k}{n}, & \text{gdy } x \in \left(\frac{k}{n}, \frac{2k+1}{2n}\right] \\ -x + \frac{k+1}{n}, & \text{gdy } x \in \left(\frac{2k+1}{2n}, \frac{k+1}{n}\right) \end{cases}$$

There holds $\mathcal{J}(u_n) \to \inf \mathcal{J}$, *but* $\inf \mathcal{J} \neq \mathcal{J}(\lim u_n)$.

Consider a function sequence $(\sin nx)$, $n \in \mathbb{N}$, defined on an interval $\Omega := (0, \frac{\pi}{2})$.

On the one hand we have

$$\int_{\Omega} \sin(nx) dx \to \int_{\Omega} 0 dx, \quad \text{as } n \to \infty,$$

while on the other

$$\int_{\Omega} \sin^2(nx) dx \to \tfrac{\pi}{4} \neq \int_{\Omega} 0^2 dx.$$

More generally:

- $\mathbb{R}^d \supset \Omega$ – nonempty, bounded open set of Lebesgue measure $M > 0$;

- $K \subset \mathbb{R}^l$ – compact;

- (f_n) – a sequence of functions from Ω to K, convergent to some function f_0 weakly*;

- φ – an arbitrary continuous real valued function on $\mathbb{R}^l$.

Consider now a sequence $(\varphi(f_n))$.

Then there exists a subsequence of $(\varphi(f_n))$ weakly* convergent to some function g. However, in general not only

$$g \neq \varphi((f_0)),$$

but g is not a function with domain in $\mathbb{R}^l$.

Theorem 1 *(Laurence Chisholm Young*, Generalized curves and the existence of an attained absolute minimum in the calculus of variations, *Comptes Rendus de la Société des Sciences et des Lettres de Varsovie, classe III, 30 (1937), 212 – 234)*

There exist:

- *subsequence of* (f_n) *(not relabeled);*
- *family* $(\nu_x)_{x\in\Omega}$ *of probability measures* , supp$\nu_x \subseteq K$,

such that $\forall\, \varphi \in C(\mathbb{R}^l)$ *the sequence* $(\varphi(f_n))$ *converges weakly** *to the function*

$$\overline{\varphi}(x) := \int\limits_{\mathbb{R}^l} \varphi(\lambda)\nu_x(d\lambda).$$

Definition 1 *(a) the family* $(\nu_x)_{x\in\Omega}$ *of probability measures is called a* Young measure associated with (or generated by) *the sequence* (f_n)*;*

(b) if the the family $(\nu_x)_{x\in\Omega}$ *does not depend on* $x \in \Omega$*, i.e. it is a 'one element family', we call it a* **homogeneous Young measure***.*

- Young measures can be looked at as elements of the space conjugate to the space

$$L^1(\Omega, C(K))$$

of Bochner integrable functions.

This approach allows us to associate Young measure with **any** Borel function $f\colon \Omega \to K$.

- It can be proved that the space $\left(L^1(\Omega, C(K))\right)^*$ is isometrically isomorphic to the space

$$L^\infty_{w^*}(\Omega; rca(K))$$

of weak* measurable mappings with values in the space of regular countably additive measures on the compact set $K \subset \mathbb{R}^l$.

- It is known, that $rca(K)$ – the space of regular, countably additive scalar measures on K, equipped with the norm $\|m\|_{rca(K)} := |m|(\Omega)$, where $|\cdot|$ stands in this case for the total variation of the measure m, is a Banach space.

- Young measures are those of these mappings from $L^\infty_{w^*}(\Omega; rca(K))$, whose values lie in the unit sphere $rca^1(K)$ of $rca(K)$.

Weak convergence of functions

By definition, a sequence $(f_n) \subset L^1_\rho(X)$ converges **weakly** to some function $f \in L^1_\rho(X)$ if and only if $\forall g \in L^\infty_\rho(X)$ there holds

$$\lim_{n\to\infty} \int\limits_X g(x) f_n(x) d\rho = \int\limits_X g(x) f(x) d\rho.$$

Theorem 2 *(Jean Dieudonné, 1957) let X be a locally compact Hausdorff space and $(X, \mathcal{A}, \rho)$ – a measure space.*
A sequence $(f_n) \subset L^1_\rho(X)$ converges weakly to some function $f \in L^1_\rho(X)$ if and only if $\forall A \in \mathcal{A}$ the limit

$$\lim_{n\to\infty} \int\limits_A f_n d\rho$$

exists and is finite.

Weak convergence of measures

Let Y be a Banach space. By definition, a sequence $(y_n) \subset Y$ converges **weakly** to some $y \in Y$, if $\forall\, w \in Y^*$ there holds

$$\lim_{n\to\infty} w(y_n) = w(y).$$

Theorem 3 *Let X be a locally compact Hausdorff space and $(X, \mathcal{B}(X))$ – a measurable space. A sequence (μ_n) of scalar measures on $\mathcal{B}(X)$ converges weakly (in the Banach space $rca(K)$) to some measure μ on $\mathcal{B}(X)$ if and only if $\forall A \in \mathcal{B}(X)$ the limit*

$$\lim_{n\to\infty} \mu_n(A)$$

exists and is finite.

Corollary 1 *Let* (μ_n) *be a sequence in the Banach space* $rca(K)$, *with elements having respective densities* g_n, $n \in \mathbb{N}$. *Then the weak convergence of* (μ_n) *(in the Banach space* $rca(K)$*) is equivalent to the weak* L^1 *convergence of the sequence* (g_n) *of densities of the elements of* (μ_n).

We now use the above result to the sequences of homogeneous Young measures.

Theorem 4 *Let $\{\Omega\}$ be an open partition of Ω into n open subsets $\Omega_1, \Omega_2, \ldots, \Omega_n$ such that*

(i) the elements of $\{\Omega\}$ are pairwise disjoint;

(ii) $\bigcup_{i=1}^{n} \overline{\Omega}_i = \overline{\Omega}$, where $\overline{A}$ denotes the closure of the set A.

Let a continuous function $f\colon \Omega \to K \subset \mathbb{R}^d$, with $K := \overline{f(\Omega)}$ compact, be of the form

$$f := \sum_{i=1}^{n} f_i \chi_{\Omega_i},$$

where for each $i = 1, 2, \ldots, n$ the function f_i has continuously differentiable inverse f_i^{-1}.

Then

a Young measure associated with f is a homogeneous one and it is absolutely continuous with respect to the Lebesgue measure on K. Its density is of the form

$$g(y) = \frac{1}{M} \sum_{i=1}^{n} |J_{f_i^{-1}}(y)| \chi_{f(\Omega_i)}.$$

Consider now a sequence (f^l) of functions of the form

$$f^l := \sum_{i=1}^{n(l)} f_i^l \chi_{\Omega_i^l} ,$$

where for fixed $l \in \mathbb{N}$ the function (f^l) is as described above.

According to Theorem 4 the Young measure associated with f^l is a homogeneous one and is absolutely continuous with respect to the Lebesgue measure dy on K with density

$$g^l(y) = \frac{1}{M} \sum_{i=1}^{n(l)} |J_{(f_i^l)^{-1}}(y)|.$$

Theorem 5 *Let* (f^l) *be the sequence of functions described above and denote by* ν^l *the Young measure (with density* g^l*) associated with the function* f^l.

Then the sequence (g^l) *is weakly convergent in* $L^1(K)$ *to some function* h *if and only if the sequence* (ν^l) *is weakly convergent (in the Banach space* $rca(K)$*) to some measure* η.

Moreover,

η *is a homogeneous Young measure with density that is equal to* h *a.e. with respect to the Lebesgue measure on* K.

One-dimensional example: the Lebesgue-Stieltjes measures

Let I be an open nondegenerate interval in $\mathbb{R}$ with length M and u – a strictly monotonic differentiable real valued function on I. We can assume that u is strictly increasing.

Then for any $\beta \in C(K, \mathbb{R})$ we have

$$\int_K \beta(y) \tfrac{1}{M}(u^{-1})'(y)dy = \int_K \beta(y) d\tfrac{1}{M}u^{-1}(y).$$

This means that the homogeneous Young measure associated with u is a Lebesgue-Stieltjes measure on K.

Let (u_n) be a sequence of real valued functions from a bounded, nondegenerate interval $I \subset \mathbb{R}$ such that:

(i) for any $n \in \mathbb{N}$ u_n is a C^∞-diffeomorphism;

(ii) $\overline{u_n^{(l)}(I)} := K_l$ is compact, $l \in \mathbb{N} \cup \{0\}$;

(iii) $u_n^{(l)}$ is strictly positive or negative, $n \in \mathbb{N}$.

Then for any fixed $l \in \mathbb{N}$ the weak L^1 convergence of the sequence $\big((u_n^{-1})^{(l)}\big)$ is equivalent to the weak convergence of the sequence of the Young-Lebesgue-Stieltjes measures associated with the elements of the sequence $\big((u_n)^{(l)}\big)$.

Thank you for attention